UNIVERSITÀ DI PAVIA

Università della Svizzera italiana

Lower bounds for the Integrality Gap of the Metric Steiner Tree Problem via a novel formulation

ISCO 2024

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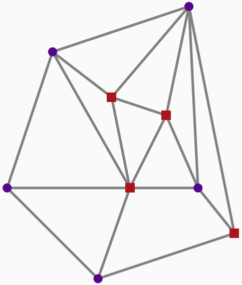
Problem definition

Steiner Tree Problem

Given an undirected, edge-weighted graph $G = (V, E)$, $|V| = n$, and a subset of vertices $T \subset V$, $|T| = t$, the Steiner Tree Problem (STP) involves finding the minimum-cost tree that spans T.

The STP is NP-Hard and the corresponding decision problem is NP-Complete [Kar10].



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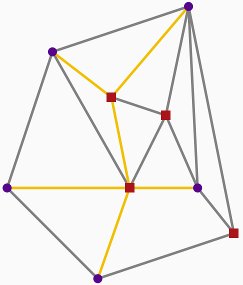

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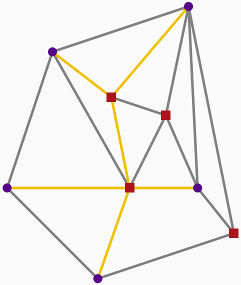

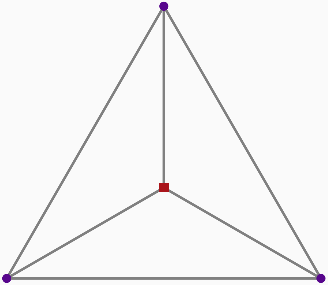

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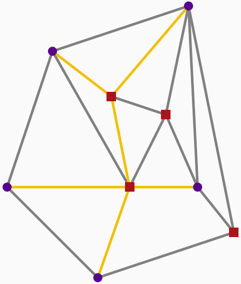

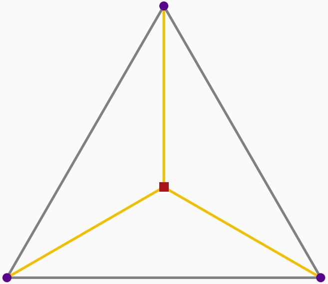

Integer Linear Programming

Bidirected cut formulation (DCUT)

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \sum_{e=\{i,j\}\in E} c_e(x_{ij}+x_{ji}) && && \text{(1a)}\\
\text{s.t.} \quad & x_{ij}+x_{ji} \le 1, && e=\{i,j\}\in E, && \text{(1b)}\\
& \sum_{(i,j)\in A\,:\,j\in W, i\notin W} x_{ij} \ge 1, && W\subset V\setminus\{r\},\ W\cap T\neq\emptyset, && \text{(1c)}\\
& x_{ij}\in\{0,1\}, && (i,j)\in A. && \text{(1d)}
\end{aligned}
$$

Integer Linear Programming

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$$\min_{\mathbf{x}} \quad \sum_{e=\{i,j\}\in E} c_e(x_{ij} + x_{ji}) \tag{1a}$$

$$\text{s.t.} \quad x_{ij} + x_{ji} \leq 1, \qquad e = \{i,j\} \in E, \tag{1b}$$

$$\sum_{(i,j)\in A\,:\, j\in W, i\notin W} x_{ij} \geq 1, \qquad W \subset V \setminus \{r\},\ W \cap T \neq \emptyset, \tag{1c}$$

$$x_{ij} \in \{0,1\}, \qquad (i,j) \in A. \tag{1d}$$

Substituiting Constraint (1d) with $0 \leq x_{ij} \leq 1$ leads to the linear relaxation of DCUT, which we denote with RDCUT. The constraints of RDUCT define the polytope $P_{\text{DCUT}}(n,t)$.

Measuring integerness

Definition 1 (Integrality gap).

The integrality gap is defined as the supremum on all the instances of the ration between the optimal integer value and the linear relaxation:

$$\alpha_{\mathsf{DCUT}}(n,t) = \sup_{\substack{G=(V,E)\in \mathsf{STP} \\ |V|=n, |T|=t}} \frac{\mathsf{DCUT}(G)}{\mathsf{RDCUT}(G)}.$$

We have $36/31 \leq \alpha_{\mathsf{DCUT}} \leq 2$ [BGRS13].

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We have $36/31 \leq \alpha_{\mathrm{DCUT}} \leq 2$ [BGRS13].

Theorem 1.

Given G' the metric closure of G, we have that

$$\frac{\mathrm{DCUT}(G)}{\mathrm{RDCUT}(G)} = \frac{\mathrm{DCUT}(G')}{\mathrm{RDCUT}(G')}.$$

Vertices

One can study the integrality gap through vertices [BB08].

The gap problem

Given $\bar{x}$ vertex of $P_{\mathrm{DCUT}}(n,t)$, we define

$$
\begin{aligned}
\frac{1}{Gap(\bar{x})} = \min\ & c^T\bar{x} \\
\text{s.t. }\ & 0 \le c_{ij} \le c_{ik} + c_{jk}, && \forall \{i,j\},\{i,k\},\{j,k\} \in E, \\
& \mathrm{DCUT}(c) \ge 1,\ \text{Duality constraints, Slackness compatibility conditions.}
\end{aligned}
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\end{aligned}
$$

Equivalence

$$
\sup_{\substack{\bar{x} \text{ vertex of} \\ P_{\mathrm{DCUT}}(n,t)}} Gap(\bar{x}) = \sup_{\substack{G=(V,E)\in \mathrm{STP} \\ |V|=n,\, |T|=t \\ G \text{ metric and complete}}} \frac{\mathrm{DCUT}(G)}{\mathrm{RDCUT}(G)}.
$$

Vertices problem

For the vertex enumeration, a software like Polymake [AGH+17] can be used. The pipeline described before for the gap problem is then executed as in [VGMG23].

Vertices problem

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n	t	time for vertices generation	# feas problems	gap
4	3	0.04	70/256	1.000
5	3	4563.57	3655/28345	1.000
5	4	2798.17	3645/24297	1.000

Table: Results obtained via Polymake for the DCUT formulation. Number of nodes, number of terminals, time for generating the vertices, number of feasible problems and maximum gap found.

A novel formulation

Complete metric (CM) formulation

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \sum_{e=\{i,j\}\in E} c_e(x_{ij} + x_{ji}) && && \text{(2a)}\\
\text{s.t.} \quad & x_{ij} + x_{ji} \leq 1, && e = \{i,j\} \in E, && \text{(2b)}\\
& \sum_{\substack{(i,j)\in A\\ j\in W, i\notin W}} x_{ij} \geq 1, && W \subset V \setminus \{r\},\ W \cap T \neq \emptyset, && \text{(2c)}\\
& \sum_{i\neq r} x_{ir} \leq 0, && && \text{(2d)}\\
& \sum_{i\neq j} x_{ij} \leq 1, && j \in V \setminus \{r\}, && \text{(2e)}\\
& 2 \cdot \sum_{i\neq j} x_{ij} - \sum_{k\neq j} x_{jk} \leq 0, && j \in V \setminus T, && \text{(2f)}\\
& x_{ij} \in \{0,1\}, && (i,j) \in A. && \text{(2g)}
\end{aligned}
$$

Properties

n	t	time for vertices generation	# feas problems	gap
4	3	0.732	4/4	1.000
5	3	44.62	5/5	1.000
5	4	37.01	44/44	1.000

Lemma 1.

It holds that the integrality gap of the CM formulation is a lower bound for the integrality gap of the DCUT formulation, i.e.,

$$\alpha_{\mathrm{CM}}(n,t) \leq \alpha_{\mathrm{DCUT}}(n,t).$$

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Moreover, we were able to prove interesting results for the CM formulation, both for integer and fractional vertices, regarding connectedness, number of edges, constraint reduction, and vertex redundancy.

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It holds that the integrality gap of the CM formulation is a lower bound for the integrality gap of the DCUT formulation, i.e.,

$$\alpha_{\text{CM}}(n,t) \leq \alpha_{\text{DCUT}}(n,t).$$

Theorem 2.

Let x be an integer point of $P_{\text{CM}}(n,t)$. Then x is an optimal solution for the CM formulation with the metric cost $c_{ij} = 2 - (x_{ij} + x_{ji}) \in \{1, 2\}$.

Algorithm One-Two-Costs (OTC) heuristic

$\mathbb{G} = \{G = (V, E) \mid G \text{ connected}, \ |V| = n, \ n \leq |E| \leq n \cdot t - t^2\}$ [MP14]
$\mathbb{T} = \{T \mid T \subset \{1, \ldots, n\}, \ |T| = t\}$
$\mathfrak{G}, \mathcal{V} = \emptyset$
for $G \in \mathbb{G}, \ T \in \mathbb{T}, \ r \in T$ **do**
    $G_{T,r}$ = node-colored graph with G as its support graph, r colored as `root`, i colored as `terminal` $\forall i \in T \setminus \{r\}$, j colored as `steiner` $\forall i \notin T$
    if $H \ncong G_{T,r} \ \forall H \in \mathfrak{G}$ **then**
        add $G_{T,r}$ to $\mathfrak{G}$
    end if
end for
for $G_{T,r} \in \mathfrak{G}$ **do**
    obtain the STP instance (G, T, r) from $G_{T,r}$ with $c_{ij} = 1$ if $\{i, j\} \in G_{T,r}$ and $c_{ij} = 2$ otherwise; solve (2a) - (2f)
    if a solution x is found and it is a non-integer vertex of $P_{\mathrm{CM}}(n, t)$ **then**
        add x to $\mathcal{V}$
    end if
end for

Special vertices

Pure half-integer vertices

Given x a non-integer vertex of $P_{CM}(n,t)$, we say that x is

- half integer (HI) if $x_{ij} \in \{0, 1/2, 1\}$ $\forall (i,j) \in A$,
- pure half integer (PHI) if $x_{ij} \in \{0, 1/2\}$ $\forall (i,j) \in A$.

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- pure half integer (PHI) if $x_{ij} \in \{0, 1/2\}\ \forall (i,j) \in A$.

Theorem 3 (PHI theorem).

Let x be a PHI vertex of $P_{\text{CM}}(n,t)$, $t \geq 3$, and let it also be a vertex of $P_{\text{DCUT}}(n,t)$ optimum for a metric cost. Suppose that $x \ncong y$ for every y vertex of $P_{\text{CM}}(n-1,t)$. Define G_x as the support graph of x. In the hypothesis that the indegree of every non-terminal node in G_x is exactly 1, the followings hold:

- the indegree of every terminal in G_x is exactly 2;
- G_x is a connected graph with n nodes;
- G_x has exactly $n+t-2$ edges.

Algorithm PHI heuristic

$\mathbb{G} = \{G = (V, E) \mid G \text{ connected},\ deg(i) \geq 2\, \forall i \in V,\ |V| = n,\ |E| = n + t - 2\}$

di$\mathbb{G}, \mathcal{V} = \emptyset$

for $G = (V, E) \in \mathbb{G}$ **do**

  if $|\{i \in V \mid deg(i) = 2\}| \leq t$ **then**

    add to di$\mathbb{G}$ every non-isomorphic orientation of G s.t. every edge can be oriented in only one way and every node has a maximum indegree of 2

  end if

end for

for di$G = (V, A) \in$ di$\mathbb{G}$ **do**

  $x_{ij} = 1/2$ iff $(i, j) \in A$ is a solution of $P_{\mathrm{CM}}(n, t)$ with

    $\cdot\ \{r\} = \{i \in V \mid indeg(i) = 0\}$

    $\cdot\ V \setminus T = \{i \in V \mid indeg(i) = 1\}$

    $\cdot\ T \setminus \{r\} = \{i \in V \mid indeg(i) = 2\}$

  if x is a feasible vertex of $P_{\mathrm{CM}}(n, t)$ **then**

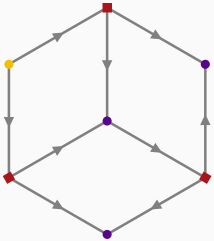
    add x to $\mathcal{V}$

  end if

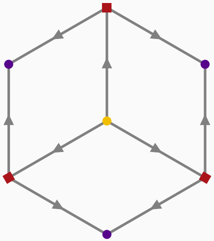
end for

		PHI			OTC		
n	t	# vert.	max gap	# vert. max. gap	# vert.	Gap	# vert. max. gap
	3	0	-	-	0	-	-
6	4	1	1/1	1	0	-	-
	5	7	1/1	7	0	-	-
	3	0	-	-	0	–	
7	4	2	10/9	2	11	10/9	2
	5	46	1/1	46	19	1/1	19
	6	71	1/1	71	8	1/1	8
	3	0	-	-	0	-	-
	4	0	-	-	19	10/9	2
8	5	89	12/11	15	195	10/9	14
	6	1070	1/1	1070	239	1/1	239
	7	758	1/1	758	0	-	-

Some vertices



$(n, t) = (7, 4)$, gap $= 10/9$.



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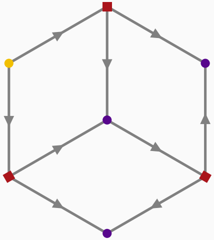

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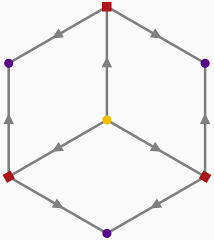

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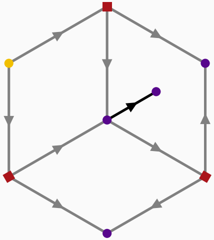
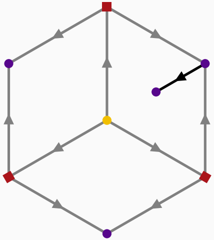


$(n, t) = (8, 5)$, gap $= 10/9$.



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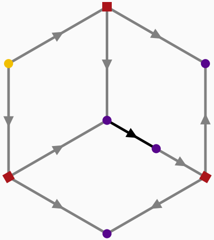

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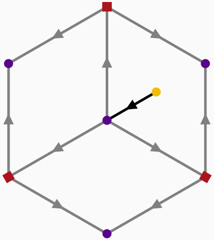

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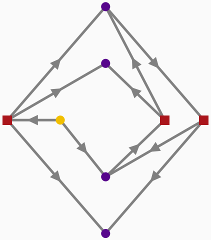
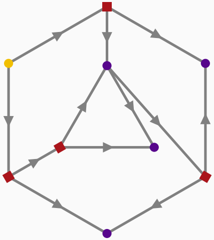
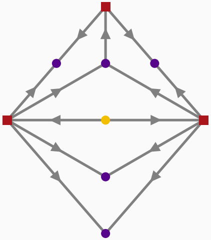
Some more vertices



$(n, t) = (8, 5)$, gap $= 12/11$.

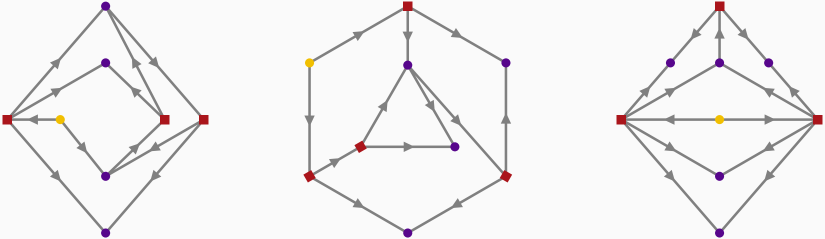


$(n, t) = (9, 5)$, gap $= 10/9$.



$(n, t) = (9, 6)$, gap $= 14/13$.

Some more vertices



$(n, t) = (8, 5)$, gap $= 12/11$. $(n, t) = (9, 5)$, gap $= 10/9$. $(n, t) = (9, 6)$, gap $= 14/13$.

- All the values of integrality gap found by PHI heuristic: $10/9$, $12/11$, $14/13$, $16/15$, $18/17$, $20/19$, $22/21$, $24/23$.
- Note how the PHI(n, t) can be generalized to vertex attaining values in the set $\{0, 1/m\}$ just by changing some parameters.

Conclusions and future work

- A novel and stricter formulation with some interesting properties.
- A problem regarding the maximization of the integrality gap.
- Two heuristics for generating vertices.

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- A novel and stricter formulation with some interesting properties.
- A problem regarding the maximization of the integrality gap.
- Two heuristics for generating vertices.

- Exploit the OTC heuristic, adding some constraints derived from numerical experiments or theoretical results.
- Same as above but with the PHI heuristic.
- Prove or disprove some conjectures we made along the way.

Fin.

For other things I do → ambrogiomb.github.io

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Properties

Lemma 2.

The support graph of any feasible point of RCM is a connected graph.

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Lemma 3.

Let x be a feasible solution for the CM formulation for a graph with $|V| = n$ nodes and $|T| = t$ terminals. Then x verifies

$$\sum_{i,j} x_{ij} \leq \min(n-1, 2t-3). \tag{3}$$

Let $t \leq \frac{n}{2} + 1$ and so $\min(n-1, 2t-3) = 2t-3$. Then, our solution is a tree with at most $2t-3$ edges, so it has $2t-3+1 = 2t-2$ nodes, with $t-2$ being Steiner vertices. Thus, it suffices to write Constraints (2c) only for

$$W = W_1 \sqcup W_2, \quad W_1 \subset T \setminus r, \; |W_1| \geq 1, \quad W_2 \subset V \setminus T, \; |W_2| \leq t-2, \tag{4}$$

instead of writing it for any $W = W_1 \sqcup W_2$, $W_2 \subset V \setminus T$.

Avoid redundancy

Lemma 4.

Let x be a vertex of $P_{\mathrm{CM}}(n,t)$. Then

$$y_{ij} = \begin{cases} x_{ij}, & \text{if } i,j \neq n+1, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

is a vertex of $P_{\mathrm{CM}}(n+1,t)$.

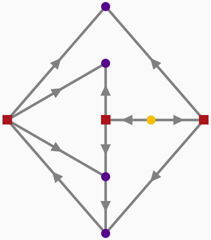
Lemma 5.

Let y be a vertex of $P_{\mathrm{CM}}(n,t)$ of the form

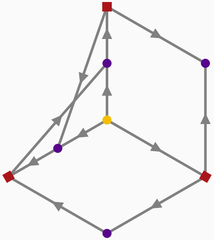
$$y_{ij} = \begin{cases} x_{ij}, & \text{if } i \neq k \neq j, \\ 0, & \text{else,} \end{cases} \tag{6}$$

for a certain $k \in V \setminus T$. Then x is a vertex of $P_{\mathrm{CM}}(n \setminus \{k\}, t) \cong P_{\mathrm{CM}}(n-1,t)$.

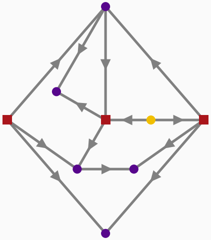
Other vertices



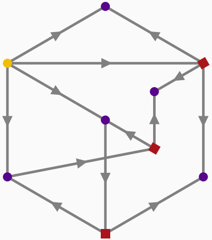
$(n, t) = (8, 5)$, gap $= 14/13$.



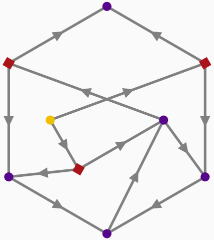
$(n, t) = (8, 5)$, gap $= 18/17$.



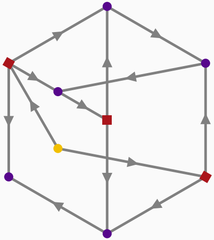
$(n, t) = (9, 6)$, gap $= 16/15$.



$(n, t) = (9, 6)$, gap $= 20/19$.

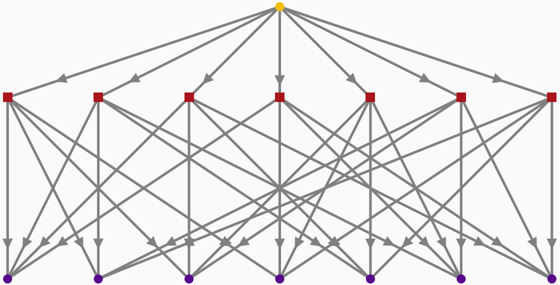


$(n, t) = (9, 6)$, gap $= 22/21$.



$(n, t) = (9, 6)$, gap $= 24/23$.

PHI generalization



If we have $x_{ij} \in \{0, 1/m\}$, the indegree of the terminal nodes must now be m, while the indegree of the Steiner nodes is again 1. This gives us $n + (m-1) \times t - m$ edges. In addition, every node has degree at least $\min(3, m)$; if $m > 3$ the number of nodes with degree 3 is at most $n - t$; there must exist one node of indegree 0, $n - t$ nodes of indegree 1, and $t - 1$ nodes of indegree m. We would have been able to find the vertex above of gap 8/7. [KPT11]