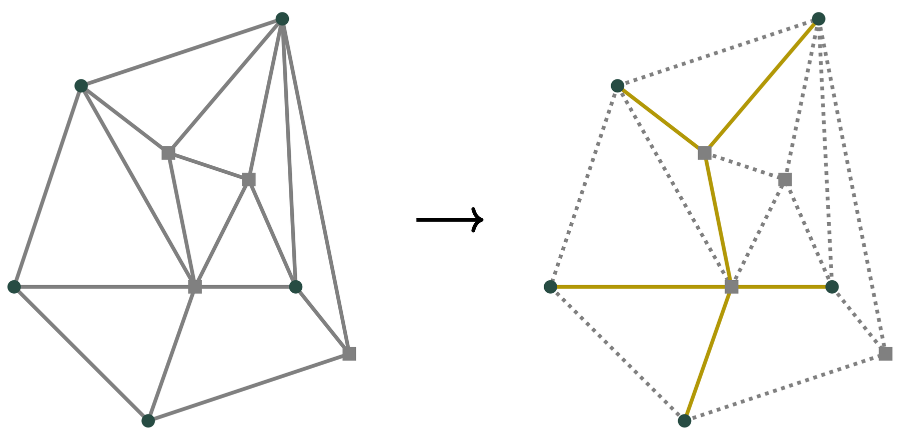
Lower bounds for the Integrality Gap of the Metric Steiner Tree Problem via a novel formulation

Ambrogio Maria Bernardelli[1] Eleonora Vercesi[2, 4]
Luca Maria Gambardella[2, 4] Stefano Gualandi[1] Monaldo Mastrolilli[2, 3]
[1]University of Pavia, Pavia 27100, Italy [2]IDSIA (USI - SUPSI), [3]SUPSI, [4]USI, Lugano 6900, Switzerland

Steiner Tree Problem (STP)

- $G = (V, E)$ undirected, edge-weighted, connected graph, $|V| = n$, $T \subset V$, $|T| = t$, $2 \leq t \leq n$.
- Find the minimum-cost tree that spans T.
- The STP is NP-Hard and the corresponding decision problem is NP-Complete [3].

Figure 1. Example of an STP instance and its solution. Green dots represent the *terminal nodes* T while gray squares represents the *potential Steiner nodes* $V \setminus T$. Potential Steiner nodes are called *Steiner nodes* of a given solution if they are part of that solution.



Bidirected cut formulation (DCUT)

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^{2\times|E|}} \quad & \sum_{e=\{i,j\}\in E} c_e(x_{ij} + x_{ji}) && && \text{(1a)}\\
\text{s.t.} \quad & x_{ij} + x_{ji} \leq 1, && \{i,j\} \in E, && \text{(1b)}\\
& x(\delta^-(W)) \geq 1, && W \subset V \setminus \{r\},\ |W \cap T| \geq 1, && \text{(1c)}\\
& x_{ij} \in \{0,1\}, && (i,j) \in A, && \text{(1d)}
\end{aligned}
$$

with $r \in T$ and A the set of arcs.
$P_{DCUT}(n,t) = \{x \mid \text{(1b), (1c)},\ 0 \leq x_{ij} \leq 1\}$.

The Gap Problem

As done in [1], given x vertex of $P_{DCUT}(n,t)$ we define

$$
\begin{aligned}
\frac{1}{\mathrm{Gap}(x)} = \min_{c\in\mathbb{R}^{|E|}} \ & c \cdot x\\
\text{s.t. } & c \text{ pseudometric},\ \mathrm{STP}(c) \geq 1,\\
& \text{duality constraints},\\
& \text{complementary slackness conditions.}
\end{aligned}
$$

The integrality gap of thet DCUT formulation is known to lie between 36/31 and 2 [2, 4].

Lemma (Metric closure gap)

Let G be an STP instance and let G^* be its metric closure. Then

$$\mathrm{Gap}(G) = \mathrm{Gap}(G^*),$$

where $\mathrm{Gap}(G)$ is the ratio between the optimal value of the STP over the instance G and the optimum of the linear relaxation of the STP formulation.

Complete Metric Formulation (CM)

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^{2\times|E|}} \quad & \sum_{\{i,j\}\in E} c_e(x_{ij} + x_{ji}) && && \text{(2a)}\\
\text{s.t.} \quad & x_{ij} + x_{ji} \leq 1, && e = \{i,j\} \in E, && \text{(2b)}\\
& x(\delta^-(W)) \geq 1, && W \subset V \setminus \{r\},\ |W \cap T| \geq 1, && \text{(2c)}\\
& x(\delta^-(r)) = 0, && && \text{(2d)}\\
& -x(\delta^-(j)) \geq -1, && j \in V \setminus \{r\}, && \text{(2e)}\\
& x(\delta^+(j)) \geq 2x(\delta^-(j)), && j \in V \setminus T, && \text{(2f)}\\
& x_{ij} \in \{0,1\}, && (i,j) \in A. && \text{(2g)}
\end{aligned}
$$

$P_{CM}(n,t) = \{x \mid \text{(2b)} - \text{(2f)},\ 0 \leq x_{ij} \leq 1\}$.

Theorem (CM equivalence)

Let $G = (V, E)$ be a complete metric graph, $T \subset V$, $r \in T$. Then there exists an optimal solution of the DCUT formulation for the STP instance G that satisfies (2d), (2e), (2f).

Lemma (CM structure)

The support graph of every feasible point of $P_{CM}(n,t)$ is a tree. Moreover, if $t \leq 1 + n/2$, any feasible point x of $P_{CM}(n,t)$ satisfies $\sum_{i,j} x_{i,j} \leq 2t - 3$ and so we can remove some of the Constraints (2c) by only taking into account the sets $W = U_1 \sqcup U_2$, $U_1 \subset T \setminus r$, $|U_1| \geq 1$, $U_2 \subset V \setminus T$, $|U_2| \leq t - 2$.

Theorem (Dimensionality reduction)

Let x be a vertex of $P_{CM}(n,t)$, $t \leq n-1$, such that there exists a k for which $x(\delta^-(k)) + x(\delta^+(k)) = 0$. Then there exists y vertex of $P_{CM}(n-1,t)$ such that x and y are isomorphic as edge-weighted node-colored directed graphs, with thee colors representing the sets $\{r\}$, $T \setminus \{r\}$, $V \setminus T$.

Lemma (1-2-costs)

Let x be an integer point of of $P_{CM}(n,t)$. Then it is an optimum for the metric cost $c_{ij} = 2 - (x_{ij} + x_{ji})$.

Pure half-integer vertices

Given x a non-integer vertex of $P_{CM}(n,t)$, we say that x is

- half integer (HI) if $x_{ij} \in \{0, 1/2, 1\}$ for all $(i,j) \in A$,
- pure half integer (PHI) if $x_{ij} \in \{0, 1/2\}$ for all $(i,j) \in A$.

Theorem (PHI theorem)

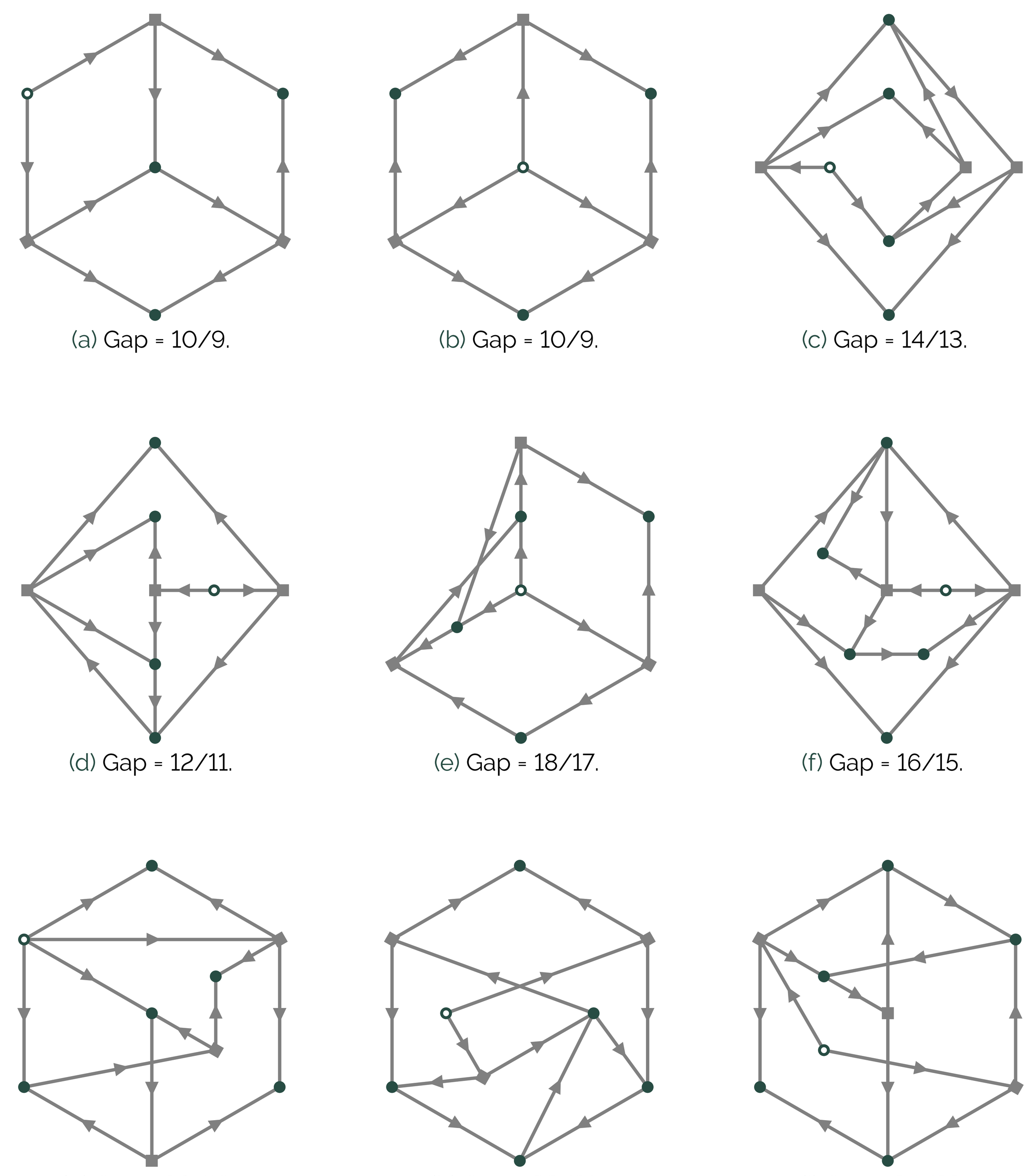
Let x be a PHI vertex of $P_{CM}(n,t)$, $t \geq 3$, and let it also be a vertex of $P_{DCUT}(n,t)$ optimum for a metric cost. Suppose that $x \not\cong y$ for every y vertex of $P_{CM}(n-1,t)$. Define G_x as the support graph of x. In the hypothesis that the indegree of every non-terminal node in G_x is exactly 1, the followings hold:

- G_x is a connected graph with n nodes;
- G_x has exactly $n + t - 2$ edges.

Pure half-integer search

```
procedure PHI(n, t)
    𝔾 = {G = (V, E) | G connected, deg(i) ≥ 2 for all i ∈ V,
          |V| = n, |E| = n + t − 2} [5]
    di𝔾 = ∅
    for G = (V, E) ∈ 𝔾 do
        if |{i ∈ V | deg(i) = 2}| ≤ t then
            add to di𝔾 every non-isomorphic orientation
            of G s.t.
             · every edge can be oriented in only one way
             · every node has a maximum indegree of 2
        end if
    end for
    𝒱 = ∅
    for diG = (V, A) ∈ di𝔾 do
        ν1, ν2, ν3 ∈ {True, False}
        ν1 = (|{i ∈ V | indeg(i) = 0}| = 1)
        ν2 = (|{i ∈ V | indeg(i) = 1}| = n − t)
        ν3 = (|{i ∈ V | indeg(i) = 2}| = t − 1)
        if ν1 ∧ ν2 ∧ ν3 then
            x_ij = 1/2 iff (i, j) ∈ A is a solution of
                  P_CM(n, t) with
                   · {r} = {i ∈ V | indeg(i) = 0}
                   · V \ T = {i ∈ V | indeg(i) = 1}
                   · T \ {r} = {i ∈ V | indeg(i) = 2}
            if x is a feasible vertex of P_CM(n, t) then
                add x to 𝒱
            end if
        end if
    end for
end procedure
```

Figure 2. Some PHI vertices of different $P_{CM}(n,t)$. Green hollow dots represent the root, green dots represent the other terminal nodes, gray squares represent Steiner nodes.



(a) Gap = 10/9. (b) Gap = 10/9. (c) Gap = 14/13.
(d) Gap = 12/11. (e) Gap = 18/17. (f) Gap = 16/15.
(g) Gap = 20/19. (h) Gap = 22/21. (i) Gap = 24/23.

Table 1. Number of vertices of $P_{CM}(n,t)$ attaining different values of integrality gap. Note that some configurations do not give interesting vertices.

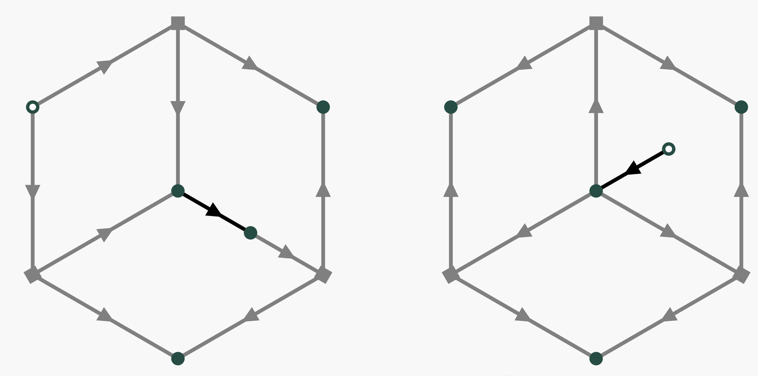
		Gap								
n	t	1	24/23	22/21	20/19	18/17	16/15	14/13	12/11	10/9
	4	0	0	0	0	0	0	0	0	2
7	5	21	0	0	0	0	0	0	0	0
	6	18	0	0	0	0	0	0	0	0
	5	40	0	0	0	2	0	7	15	0
8	6	382	0	0	0	0	0	0	0	0
	7	122	0	0	0	0	0	0	0	0
	5	6	0	0	0	0	0	9	30	12
9	6	1686	6	21	16	45	75	179	0	0
	7	4742	0	0	0	0	0	0	0	0
	8	763	0	0	0	0	0	0	0	0

Other research directions

1-2-costs heuristic

We generate all the non-isomorphic, connected, node-colored graphs. Given such a graph, we then obtain an STP instance by giving cost 1 to the edges that appear in the graph and cost 2 to all the others.

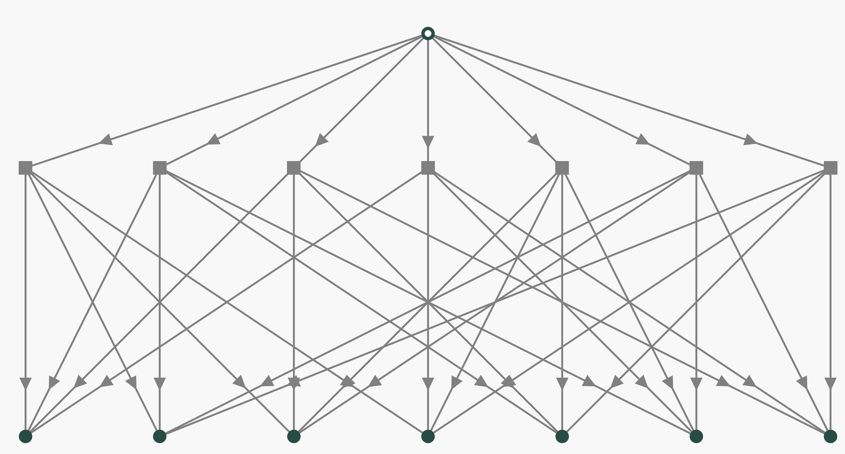
Figure 3. Some HI vertices of $P_{CM}(8,5)$. Gray lines represent a value of $x_{ij} = 1/2$ while black lines represent a value of $x_{ij} = 1$.



(a) Gap = 10/9. (b) Gap = 10/9.

Generalization fo PHI theorem

Figure 4. A vertex of $P_{CM}(15,8)$ with Gap= 8/7. Gray thin lines represent a value of $x_{ij} = 1/4$.



References

[1] Benoit, G., and Boyd, S., Finding the exact integrality gap for small traveling salesman problems. *Mathematics of Operations Research*, **33**(4), pp. 921–931, 2008.

[2] Byrka, J., and Grandoni, F., and Rothvoß, T., and Sanità, L., Steiner tree approximation via iterative randomized rounding, *Journal of the ACM*, **60**(1), pp. 1–33, 2013.

[3] Karp, R. M., Reducibility among combinatorial problems, *Springer Berlin Heidelberg*, 2010.

[4] Ljubić, I., Solving Steiner trees: Recent advances, challenges, and perspectives, *Networks*, **77**(2), pp. 177–204, 2021.

[5] McKay, B. D., and Piperino, A., Practical graph isomorphism, II, *Journal of symbolic computation*, **60**, pp. 94–112, 2014.

CompOpt group → ← arxiv preprint