Multilinear Constraints and Linearization Techniques for ACOPF Problem

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Problem definition

The problem

Let us take a network modeled as a graph $(\mathcal{B}, \mathcal{L})$, where $\mathcal{B}$ represents the set of buses and $\mathcal{L}$ represents the set of lines. For every bus k we have a (possibly empty) set of generators $\mathcal{G}(k)$ located at bus k. The problem consists of meeting the energy demand at every bus, and doing so with the lowest possible energy generation cost.

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More precisely, we have the following variables:

- for each bus k we have a complex voltage $V_k = |V_k|e^{j\delta_k}$;
- for each branch km we have two variables S_{km} and S_{mk}, the complex power injected into the branch at k and at m, respectively;
- for each generator g there is power generation $P_g^G + jQ_g^G$.

These variables are subjected to five classes of constraints.

Polar coordinates formulation

$$\inf_{\substack{P_g^G, Q_g^G, \delta_k, \\ |V_k|, S_{km}}} \sum_{g \in \mathcal{G}} F_g(P_g^G, Q_g^G) \tag{1a}$$

s.t.

AC power flow laws:

$$S_{km} = (G_{kk} - jB_{kk})|V_k|^2 + (G_{km} - jB_{km})|V_k||V_m| \cdot (\cos(\theta_{km}) + j\sin(\theta_{km})) \qquad \forall km \in \mathcal{L}, \tag{1b}$$

Flow balance constraints:

$$\sum_{km \in L} S_{km} + P_k^L + jQ_k^L = \sum_{g \in \mathcal{G}(k)} P_g^G + j \sum_{g \in \mathcal{G}(k)} Q_g^G \qquad \forall k \in \mathcal{B}, \tag{1c}$$

Branch limits, generator limits, voltage bounds:

$$|S_{km}|^2 \leq U_{km} \qquad \forall km \in \mathcal{L}, \tag{1d}$$

$$P_g^{\min} \leq P_g^G \leq P_g^{\max}, \; Q_g^{\min} \leq Q_g^G \leq Q_g^{\max} \qquad \forall g \in \mathcal{G}, \tag{1e}$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max} \qquad \forall k \in \mathcal{B}, \tag{1f}$$

$$\theta_{km}^{\min} \leq \theta_{km} \leq \theta_{km}^{\max} \qquad \forall km \in \mathcal{L}. \tag{1g}$$

Variable substitution

One can introduce auxiliary variables to tackle the problem of having sine and cosine functions:

$$
\begin{aligned}
c_{km} &= |V_k||V_m| \cdot \cos(\theta_{km}) && \forall km \in \mathcal{L},\\
s_{km} &= |V_k||V_m| \cdot \sin(\theta_{km}) && \forall km \in \mathcal{L},\\
c_{kk} &= |V_k|^2 && \forall k \in \mathcal{B}.
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Substituting such variables in the model without adding their definitions gives us a first relaxed model.

Note that by doing so we manage to remove sine and cosine functions but we also lose crucial relations between the new variables.

A first relaxed model

$$
\begin{aligned}
\inf_{\substack{P_g^G, Q_g^G, c_{km},\\ s_{km}, S_{km}, P_{km}, Q_{km}}} \quad & F(x) := \sum_{g \in \mathcal{G}} F_g(P_g^G) && && \text{(2a)}\\
\text{Subject to:} \quad & P_{km} = G_{kk} c_{kk} + G_{km} c_{km} + B_{km} s_{km} && \forall km \in \mathcal{L}, && \text{(2b)}\\
& Q_{km} = -B_{kk} c_{kk} - B_{km} c_{km} + G_{km} s_{km} && \forall km \in \mathcal{L}, && \text{(2c)}\\
& S_{km} = P_{km} + jQ_{km} && \forall km \in \mathcal{L}, && \text{(2d)}\\
& \sum_{km \in L} S_{km} + P_k^L + jQ_k^L = \sum_{g \in \mathcal{G}(k)} P_g^G + j \sum_{g \in \mathcal{G}(k)} Q_g^G && \forall k \in \mathcal{B}, && \text{(2e)}\\
& P_{km}^2 + Q_{km}^2 \le U_{km} && \forall km \in \mathcal{L}, && \text{(2f)}\\
& {V_k^{\min}}^2 \le c_{kk} \le {V_k^{\max}}^2 && \forall k \in \mathcal{B}, && \text{(2g)}\\
& P_g^{\min} \le P_g^G \le P_g^{\max},\ Q_g^{\min} \le Q_g^G \le Q_g^{\max} && \forall g \in \mathcal{G}, && \text{(2h)}\\
& c_{kk} \ge 0 && \forall k \in \mathcal{B}, && \text{(2i)}\\
& V_k^{\max} V_m^{\max} \ge c_{km} \ge 0 && \forall km \in \mathcal{L}, && \text{(2j)}\\
& -V_k^{\max} V_m^{\max} \le s_{km} \le V_k^{\max} V_m^{\max} && \forall km \in \mathcal{L}, && \text{(2k)}\\
& c_{km} = c_{mk},\ s_{km} = -s_{mk} && \forall km \in \mathcal{L}. && \text{(2l)}
\end{aligned}
$$

Jabr

Equality

To link the c and s variables we make use of the following equality:

$$c_{km}^2 + s_{mk}^2 = c_{kk} c_{mm} \quad \forall km \in \mathcal{L}. \tag{3}$$

We will denote by Jabr equality ACOPF relaxation the model (2) together with constraints (3).

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These nonconvex couplings constraints can be relaxed as follows.

Inequality

$$c_{km}^2 + s_{mk}^2 \leq c_{kk} c_{mm} \quad \forall km \in \mathcal{L}. \tag{4}$$

Trees and cycles

Trees

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Lemma 1.

If $(\mathcal{B}, \mathcal{L})$ is a multisource radial network, then the Jabr equality ACOPF relaxation is exact[Jab06].

[Jab06] Rabih A. Jabr. "Radial distribution load flow using conic programming". In: *IEEE transactions on power systems* 21.3 (2006), pp. 1458–1459

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Why do we need a tree structure for the exactness of the model?

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Loop constraints

Definition 1 (Loop constraint).

Given a cycle $\mathcal{C}$ on nodes $\{k_1, \dots, k_n\}$, we define the loop constraint on $\mathcal{C}$ as the following

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subset [n] \\ |A| = 2j}} (-1)^j \prod_{h \in A} s_{k_h k_{h+1}} \prod_{l \in A^c} c_{k_l k_{l+1}} = \prod_{i=1}^{n} c_{k_i k_i} \tag{5}$$

with $A^c := [n] \setminus A$.

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with $A^c := [n] \setminus A$.

Lemma 2.

The Jabr equality ACOPF relaxation together with the additional loop constraint (5) written for every cycle of $(\mathcal{B}, \mathcal{L})$ is exact[KDS16].

[KDS16] Burak Kocuk, Santanu S. Dey, and X. Andy Sun. "Strong SOCP relaxations for the optimal power flow problem". In: *Operations Research* 64.6 (2016), pp. 1177–1196

Constraint redundancy

Definition 2 (Cycle space).

The (binary) cycle space of an undirected graph is the set of its even-degree subgraphs.

Definition 3 (Cycle basis).

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Lemma 3.

It is sufficient to write (5) for every cycle in a cycle basis of $(\mathcal{B}, \mathcal{L})$.

Some Linearizations

Multilinear (I)

A first idea is to use cycles of length three and four in order to have polynomials of degree 3 and 4, that can be replaced exactly by two bilinear constraints[KDS16].

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We instead asked ourself if it was possible to find good linear approximations for multilinear polynomials in general.

Let $F = \sum_{I \in \mathcal{I}} \prod_{v \in I} x_v$ be a multilinear polinomial and let us focus on a single monomial $f = \prod_{v \in I} x_v$, with $x_v \in [l_v, u_v]$. Define the cuboid $\mathfrak{C} := \prod_{v \in I} [l_v, u_v]$. We are looking for valid linear inequalities, that is, hyperplanes π such that either $\pi(x) \geq f(x)$ for all $x \in \mathfrak{C}$ or $\pi(x) \leq f(x)$ for all $x \in \mathfrak{C}$.

[KDS16] Burak Kocuk, Santanu S. Dey, and X. Andy Sun. "Strong SOCP relaxations for the optimal power flow problem". In: *Operations Research* 64.6 (2016), pp. 1177–1196

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In addition, we are looking for "good" hyperplanes, that is, we would like $\pi(x^i) = f(x^i)$ for some $x^i \in \mathfrak{C}$.

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Lemma 4.

The hyperplane $\pi_a(x) := \prod_{v \in I} a_v + \sum_{v \in I} C_v(x_v - a_v)$, where $C_v := \prod_{v' \in I \setminus \{v\}} a_v$, is the only hyperplane such that $\pi(a) = f(a)$ and $\pi(y) = \prod_{v \in I} y_v$ for all vertices y in $\mathfrak{C}$, adjacent to a.

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Not all of these hyperplanes are separating hyperplanes. We proved the following results that completely characterize them.

Multilinear (III)

Theorem 1.

The hyperplane $\pi := \pi_a$ is a separating hyperplane if and only if either for all $J \subset I$, $J = \{j_1, \dots, j_s\}$, $k = 1, \dots, s-1$ and $J_k := \{j_1, \dots, j_k\}$, defining $x^{J_k} := a + \sum_{v \in J_k} d_v$, the following holds:

$$\sum_{k=2}^{s} \left(\prod_{\substack{v \in I \\ v \neq j_k}} a_v - \prod_{\substack{v \in I \\ v \neq j_k}} x_v^{J_k} \right) (a_{j_k}^{op} - a_{j_k}) \geq 0 \tag{6}$$

or for all $J \subset I$, $J = \{j_1, \dots, j_s\}$:

$$\sum_{k=2}^{s} \left(\prod_{\substack{v \in I \\ v \neq j_k}} a_v - \prod_{\substack{v \in I \\ v \neq j_k}} x_v^{J_k} \right) (a_{j_k}^{op} - a_{j_k}) \leq 0. \tag{7}$$

Convex approximations

However, the number of variables introduced by this approach grows exponentially with respect to the number of edges in the chosen cycle basis of the graph.

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For this reason, we explore a different approach: we reintroduce the variables corresponding to the phase angles θ_{km} and link them to the variable c_{km} through a convex relaxation of the constraints

$$
\begin{aligned}
c_{km} &= |V_k||V_m| \cdot \cos(\theta_{km}), \\
s_{km} &= |V_k||V_m| \cdot \sin(\theta_{km}), \\
c_{kk} &= |V_k|^2.
\end{aligned}
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OBBT

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Then, we applied the multilinear techniques presented before to the now–trilinear constraints

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$$s_{km} = |V_k||V_m| \cdot \sin(\theta_{km}).$$

We then extended the multilinear term approximation to the Jabr relaxation. To enhance accuracy, we developed an Objective-Based Bound Tightening (OBBT) procedure, which progressively refines the linear approximations of trilinear terms by tightening the bounds.

Some results

Case	**LB**			**Violation**		**Time (s)**	
	CutP[BV23]	OBBT	Primal Bound	CutP	OBBT	CutP	OBBT
case9	5 296.4193	5 296.6482	5 296.6862	46.133	46.082	0.02	0.04
case30	572.9524	574.4354	576.8923	64.261	60.218	0.12	13.32
case118	129 307.2367	129 313.6874	129 358.9523	711.2868	686.787	0.71	214.11

[BV23] Daniel Bienstock and Matias Villagra. "Accurate linear cutting-plane relaxations for ACOPF". In: *arXiv preprint arXiv:2312.04251* (2023)

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We observe a modest improvement in all lower bounds, but at the cost of significantly higher computation times. Nevertheless, this drawback may not be critical for two main reasons:

- The OBBT computations can be parallelized.
- It may be unnecessary to update all variable bounds at every iteration, given that most of them are interdependent.

[BV23] Daniel Bienstock and Matias Villagra. "Accurate linear cutting-plane relaxations for ACOPF". In: *arXiv preprint arXiv:2312.04251* (2023)

Some more results

CutP

Case	Trigonometric		Voltages	Trilinear	
	sin	cos	$v^{(2)}$	$V_k V_m s_{km}$	$V_k V_m c_{km}$
case9	[-0.0136,0.0043]	[-0.9996,-0.9962]	[0.0000,0.1100]	[-0.0025,1.1720]	[-0.0012,1.2041]
case30	[-0.0217,0.0211]	[-1.0000,-0.9991]	[0.0000,0.0525]	[-0.0039,1.0991]	[-0.0000,1.1352]
case118	[-0.0590,0.0451]	[-1.0000,-0.9867]	[0.0000,0.0371]	[-0.0162,1.0525]	[-0.0027,1.1234]

OBBT

Case	Trigonometric		Voltages	Trilinear	
	sin	cos	$v^{(2)}$	$V_k V_m s_{km}$	$V_k V_m c_{km}$
case9	[-0.0003,0.0005]	[-0.0066,-0.0001]	[0.0000,0.0636]	[-0.0068,0.0101]	[0.0007,0.0078]
case30	[-0.0001,0.0000]	[-0.0031,-0.0000]	[0.0000,0.0525]	[-0.0137,0.0113]	[0.0000,0.0039]
case118	[-0.0504,0.1494]	[-0.7249,-0.0000]	[0.0000,0.0355]	[-0.0608,0.0272]	[-0.0148,0.7847]

Can we do better (relaxations)?

A natural question arises: is there a tighter convex relaxation of the multilinear right-hand side of the ACOPF constraint

$$P_k^G - P_k^L - G_{kk}|V_k|^2 = |V_k| \sum_m |V_m| (G_{km} \cos(\theta_{km}) - B_{km} \sin(\theta_{km}))$$

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than the one obtained by taking the convex hull of each monomial and summing the resulting relaxations?

By generalizing a result from the literature[GNB25], we showed that the answer to this question is negative.

[GNB25] Cheng Guo, Harsha Nagarajan, and Merve Bodur. "Tightening quadratic convex relaxations for the alternating current optimal transmission switching problem". In: *INFORMS Journal on Computing* (2025)

Conclusions & future works

We reviewed the classical Jabr relaxation and explored linearization techniques for the multilinear constraints necessary to achieve exactness. In addition, we showed preliminary experiments with an OBBT procedure.

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The next steps include the development of a Spatial Branching Algorithm, which is currently in progress, along with efforts to design a faster and more efficient OBBT procedure. Another direction we aim to pursue is a theoretical study of constraint violations with respect to the multilinear approximation. In particular, we seek to bound how far a relaxed solution can be from a feasible one.

Fine.