UNIVERSITÀ DI PAVIA

# AC Optimal Power Flow problem

## A study on Jabr relaxation

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Problem definition

Trees and cycles

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## Problem definition

# The problem

Let us take a network modeled as a graph $(\mathcal{B}, \mathcal{L})$, where $\mathcal{B}$ represents the set of buses and $\mathcal{L}$ represents the set of lines. For every bus $k$ we have a (possibly empty) set of generators $\mathcal{G}(k)$ located at bus $k$. The problem consists of meeting the energy demand at every bus, and doing so with the lowest possible energy generation cost.

# The problem

Let us take a network modeled as a graph $(\mathcal{B}, \mathcal{L})$, where $\mathcal{B}$ represents the set of buses and $\mathcal{L}$ represents the set of lines. For every bus $k$ we have a (possibly empty) set of generators $\mathcal{G}(k)$ located at bus $k$. The problem consists of meeting the energy demand at every bus, and doing so with the lowest possible energy generation cost.

More precisely, we have the following variables:

- for each bus $k$ we have a complex voltage $V_k = |V_k|e^{j\delta_k}$;
- for each branch $km$ we have two variables $S_{km}$ and $S_{mk}$, the complex power injected into the branch at $k$ and at $m$, respectively;
- for each generator $g$ there is power generation $P_g^G + jQ_g^G$.

These variables are subjected to five classes of constraints.

# Polar coordinates formulation

$$\inf_{\substack{P_g^G, Q_g^G, \delta_k, \\ |V_k|, S_{km}}} \sum_{g \in \mathcal{G}} F_g(P_g^G, Q_g^G) \tag{1a}$$

s.t.

AC power flow laws:

$$S_{km} = (G_{kk} - jB_{kk})|V_k|^2 + (G_{km} - jB_{km})|V_k||V_m| \cdot (\cos(\theta_{km}) + j\sin(\theta_{km})) \qquad \forall km \in \mathcal{L}, \tag{1b}$$

Flow balance constraints:

$$\sum_{km \in L} S_{km} + P_k^L + jQ_k^L = \sum_{g \in \mathcal{G}(k)} P_g^G + j \sum_{g \in \mathcal{G}(k)} Q_g^G \qquad \forall k \in \mathcal{B}, \tag{1c}$$

Branch limits, generator limits, voltage bounds:

$$|S_{km}|^2 \leq U_{km} \qquad \forall km \in \mathcal{L}, \tag{1d}$$

$$P_g^{\min} \leq P_g^G \leq P_g^{\max},\ Q_g^{\min} \leq Q_g^G \leq Q_g^{\max} \qquad \forall g \in \mathcal{G}, \tag{1e}$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max} \qquad \forall k \in \mathcal{B}, \tag{1f}$$

$$\theta_{km}^{\min} \leq \theta_{km} \leq \theta_{km}^{\max} \qquad \forall km \in \mathcal{L}. \tag{1g}$$

# Variable substitution

One can introduce auxiliary variables to tackle the problem of having sine and cosine functions:

$$
\begin{aligned}
c_{km} &= |V_k||V_m| \cdot \cos(\theta_{km}) && \forall km \in \mathcal{L}, \\
s_{km} &= |V_k||V_m| \cdot \sin(\theta_{km}) && \forall km \in \mathcal{L}, \\
c_{kk} &= |V_k|^2 && \forall k \in \mathcal{B}.
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$$

Substituting such variables in the model without adding their definitions gives us a first relaxed model.

Note that by doing so we manage to remove sine and cosine functions but we also lose crucial relations between the new variables.

# A first relaxed model

$$
\begin{aligned}
\inf_{\substack{P_g^G, Q_g^G, c_{km},\\ s_{km}, S_{km}, P_{km}, Q_{km}}} \quad & F(x) := \sum_{g \in \mathcal{G}} F_g(P_g^G) && && (2a)\\
\text{Subject to:} \quad & P_{km} = G_{kk}c_{kk} + G_{km}c_{km} + B_{km}s_{km} && \forall km \in \mathcal{L}, && (2b)\\
& Q_{km} = -B_{kk}c_{kk} - B_{km}c_{km} + G_{km}s_{km} && \forall km \in \mathcal{L}, && (2c)\\
& S_{km} = P_{km} + jQ_{km} && \forall km \in \mathcal{L}, && (2d)\\
& \sum_{km \in L} S_{km} + P_k^L + jQ_k^L = \sum_{g \in \mathcal{G}(k)} P_g^G + j \sum_{g \in \mathcal{G}(k)} Q_g^G && \forall k \in \mathcal{B}, && (2e)\\
& P_{km}^2 + Q_{km}^2 \leq U_{km} && \forall km \in \mathcal{L}, && (2f)\\
& {V_k^{\min}}^2 \leq c_{kk} \leq {V_k^{\max}}^2 && \forall k \in \mathcal{B}, && (2g)\\
& P_g^{\min} \leq P_g^G \leq P_g^{\max},\ Q_g^{\min} \leq Q_g^G \leq Q_g^{\max} && \forall g \in \mathcal{G}, && (2h)\\
& c_{kk} \geq 0 && \forall k \in \mathcal{B}, && (2i)\\
& V_k^{\max} V_m^{\max} \geq c_{km} \geq 0 && \forall km \in \mathcal{L}, && (2j)\\
& -V_k^{\max} V_m^{\max} \leq s_{km} \leq V_k^{\max} V_m^{\max} && \forall km \in \mathcal{L}, && (2k)\\
& c_{km} = c_{mk},\ s_{km} = -s_{mk} && \forall km \in \mathcal{L}. && (2l)
\end{aligned}
$$

# Jabr

## Equality

To link the $c$ and $s$ variables we make use of the following equality:

$$c_{km}^2 + s_{mk}^2 = c_{kk} c_{mm} \quad \forall km \in \mathcal{L}. \tag{3}$$

We will denote by Jabr equality ACOPF relaxation the model (2) together with constraints (3).

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These nonconvex couplings constraints can be relaxed as follows.

## Inequality

$$c_{km}^2 + s_{mk}^2 \leq c_{kk} c_{mm} \quad \forall km \in \mathcal{L}. \tag{4}$$

# Trees and cycles

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**Lemma 1.**

If $(\mathcal{B}, \mathcal{L})$ is a multisource radial network, then the Jabr equality ACOPF relaxation is exact[1].

[1] Rabih A Jabr. "Radial distribution load flow using conic programming". In: *IEEE transactions on power systems* 21.3 (2006), pp. 1458–1459

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Why do we need a tree structure for the exactness of the model?

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# Loop constraints

**Definition 1 (Loop constraint).**

Given a cycle $\mathcal{C}$ on nodes $\{k_1, \ldots, k_n\}$, we define the loop constraint on $\mathcal{C}$ as the following

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subset [n] \\ |A|=2j}} (-1)^j \prod_{h \in A} s_{k_h k_{h+1}} \prod_{l \in A^c} c_{k_l k_{l+1}} = \prod_{i=1}^{n} c_{k_i k_i}, \tag{5}$$

with $A^c := [n] \setminus A$.

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with $A^c := [n] \setminus A$.

**Lemma 2.**

The Jabr equality ACOPF relaxation together with the additional loop constraint (5) written for every cycle of $(\mathcal{B}, \mathcal{L})$ is exact[2].

[2] Burak Kocuk, Santanu S. Dey, and X. Andy Sun. "Strong SOCP relaxations for the optimal power flow problem". In: *Operations Research* 64.6 (2016), pp. 1177–1196

# Constraint redundancy

**Definition 2 (Cycle space).**

The (binary) cycle space of an undirected graph is the set of its even-degree subgraphs.

**Definition 3 (Cycle basis).**

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**Lemma 3.**

It is sufficient to write (5) for every cycle in a cycle basis of $(\mathcal{B}, \mathcal{L})$.

# Some Linearizations

# Multilinear (I)

A first idea is to use cycles of length three and four in order to have polynomials of degree 3 and 4, that can be replaced exactly by two bilinear constraints[3].

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We instead asked ourself if it was possible to find good linear approximations for multilinear polynomials in general.

Let $F = \sum_{I \in \mathcal{I}} \prod_{v \in I} x_v$ be a multilinear polinomial and let us focus on a single monomial $f = \prod_{v \in I} x_v$, with $x_v \in [l_v, u_v]$. Define the cuboid $\mathfrak{C} := \prod_{v \in I} [l_v, u_v]$. We are looking for valid linear inequalities, that is, hyperplanes $\pi$ such that either $\pi(x) \geq f(x)$ for all $x \in \mathfrak{C}$ or $\pi(x) \leq f(x)$ for all $x \in \mathfrak{C}$.

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**Lemma 4.**

The hyperplane $\pi_a(x) := \prod_{v \in I} a_v + \sum_{v \in I} C_v(x_v - a_v)$, where $C_v := \prod_{v' \in I \setminus \{v\}} a_v$, is the only hyperplane such that $\pi(a) = f(a)$ and $\pi(y) = \prod_{v \in I} y_v$ for all vertices $y$ in $\mathfrak{C}$, adjacent to $a$.

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Not all of these hyperplanes are separating hyperplanes. We proved the following results that completely characterize them.

# Multilinear (III)

**Theorem 1.**

The hyperplane $\pi := \pi_a$ is a separating hyperplane if and only if either for all $J \subset I$, $J = \{j_1, \dots, j_s\}$, $k = 1, \dots, s-1$ and $J_k := \{j_1, \dots, j_k\}$, defining $x^{J_k} := a + \sum_{v \in J_k} d_v$, the following holds:

$$\sum_{k=2}^{s} \left( \prod_{\substack{v \in I \\ v \neq j_k}} a_v - \prod_{\substack{v \in I \\ v \neq j_k}} x_v^{J_k} \right) (a_{j_k}^{op} - a_{j_k}) \geq 0 \tag{6}$$

or for all $J \subset I$, $J = \{j_1, \dots, j_s\}$:

$$\sum_{k=2}^{s} \left( \prod_{\substack{v \in I \\ v \neq j_k}} a_v - \prod_{\substack{v \in I \\ v \neq j_k}} x_v^{J_k} \right) (a_{j_k}^{op} - a_{j_k}) \leq 0. \tag{7}$$

# Angle variables (I)

However, the number of variables introduced by this approach grows exponentially with respect to the number of edges in the chosen cycle basis of the graph.

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For this reason, we explore a different approach: we reintroduce the variables corresponding to the phase angles $\theta_{km}$ and link them to the variable $c_{km}$ through a convex relaxation of the constraints

$$
\begin{aligned}
c_{km} &= |V_k||V_m| \cdot \cos(\theta_{km}), \\
s_{km} &= |V_k||V_m| \cdot \sin(\theta_{km}).
\end{aligned}
$$

# Angle variables (II)

A natural question arises: is there a tighter convex relaxation of the multilinear right-hand side of the ACOPF constraint

$$P_k^G - P_k^L - G_{kk}|V_k|^2 = |V_k| \sum_m |V_m|(G_{km}\cos(\theta_{km}) - B_{km}\sin(\theta_{km}))$$

than the one obtained by taking the convex hull of each monomial and summing the resulting relaxations?

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By generalizing a result from the literature[4], we showed that the answer to this question is negative.

[4] Cheng Guo, Harsha Nagarajan, and Merve Bodur. "Tightening quadratic convex relaxations for the alternating current optimal transmission switching problem". In: *INFORMS Journal on Computing* (2025)

# Conclusions & future works

# Conclusions

- Some models and relaxations for the OPF problem.
- The study of the exactness of the relaxation with respect to the network structure.
- Some multilinear approaches and theoretical results.

# Future works

- Spatial branching using the convex relaxation of sine and cosine (under development).
- Optimization-Based Bound Tightening for the trilinear terms in order to get better approximations (under development).
- Compare not only new lower bounds but also have a numerical and theoretical study on constraints violation.

# Fine.