Maximizing Growth Rates in Finite Conway's Game of Life

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Motivation and Setting

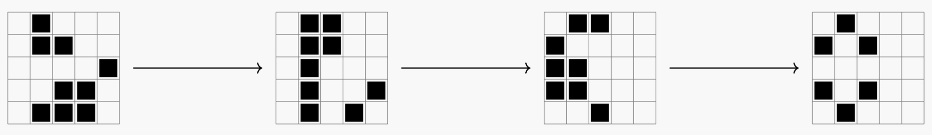
Conway's Game of Life (GoL) is a cellular automaton traditionally studied on the infinite square grid. We investigate **finite grids** with square, cylindrical, and toroidal boundary conditions, and study the following question:

How large can the population grow after T iterations?

We formulate the problem as a binary integer linear programming (ILP) model and compute optimal growth rates for small values of T and for small grid sizes.

Rules

- Any live cell with fewer than two live neighbours dies.
- Any live cell with two or three live neighbours survives to the next generation.
- Any live cell with more than three live neighbours dies.
- Any dead cell with exactly three live neighbours becomes a live cell; otherwise, it remains dead.



Growth Rate Maximization

Consider GoL on an $n \times n = n^2$ grid and let $X(T)$ denote the set of live cells at time T. Given an initial configuration $X(0)$, we define the **growth rate** after T steps as

$$g_T(X) = \frac{|X(T)| - |X(0)|}{|X(0)|}.$$

Fix an initial population size $m \in \{1, \dots, n^2\}$. We define the maximal growth rate

$$g_{T,m}(n) = \max\Big\{ g_T(X) \mid X \in \mathrm{GoL}(n),\ |X(0)| = m \Big\}.$$

Finally, we define the maximal growth rate on an n^2 grid as

$$\mathcal{G}_T(n) = \max_m g_{T,m}(n).$$

ILP Formulation

We formulate the problem as a **binary ILP**.

$$x_{i,j,t} = \begin{cases} 1 & \text{if cell } (i,j) \text{ is alive at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

The GoL update rules induce nonlinear logical relations between consecutive time steps, which we linearize using auxiliary binary variables, obtaining the following model:

$$\begin{aligned}
\Gamma_{T,m}(n) = \max \ & \sum_{(i,j)\in n^2} x_{i,j,T} \\
\text{s.t.} \ & \sum_{(i,j)\in n^2} x_{i,j,0} = m, \\
& x_{i,j,t} + \sum_{(k,l)\in a(i,j)} x_{k,l,t} \geq 3x_{i,j,t+1} && \forall (i,j,t) \in I, \\
& \sum_{(k,l)\in a(i,j)} x_{k,l,t} \leq 3 + (M_{i,j} - 2)(1 - x_{i,j,t+1}) && \forall (i,j,t) \in I, \\
& \sum_{(k,l)\in a(i,j)} x_{k,l,t} \geq 4y_{i,j,t} && \forall (i,j,t) \in I, \\
& x_{i,j,t} + \sum_{(k,l)\in a(i,j)} x_{k,l,t} \leq 2 + M_{i,j}(1 - z_{i,j,t}) && \forall (i,j,t) \in I, \\
& 1 - x_{i,j,t+1} \leq y_{i,j,t} + z_{i,j,t} && \forall (i,j,t) \in I, \\
& x_{i,j,t}, y_{i,j,t}, z_{i,j,t} \in \{0,1\} && \forall (i,j,t) \in I, \\
& x_{i,j,T} \in \{0,1\} && \forall (i,j) \in n^2.
\end{aligned}$$

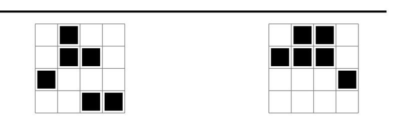
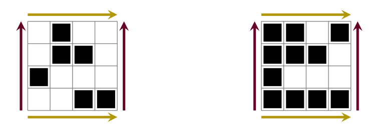
Here $I = \{(i,j,t) \mid i,j = 1,\dots,n,\ t = 0,\dots,T-1\}$, $a(i,j)$ denotes the set of neighbours of cell (i,j), and $M_{i,j} = |a(i,j)| - 1$. Note that

$$g_{T,m}(n) = -1 + \frac{1}{m}\Gamma_{T,m}(n).$$

The formulation contains $O(n^2T)$ binary variables and constraints and can be viewed as a **time-expanded** representation of the grid.

Different Topologies

Boundary conditions determine the neighbourhood of cells on the grid.

Topology	Periodicity	Examples T = 0	Examples T = 1
Square	none		
Cylinder	horizontal		
Torus	horizontal and vertical		

Quantities associated with these topologies are denoted by the superscripts s, c, and t, respectively.

Preliminary Computational Results

Using the ILP formulation implemented with `Gurobi 13.0`, we compute optimal solutions for small values of n and T. As these values increase, the ILP quickly becomes harder to solve for some values of m, making larger instances **computationally challenging**. The table was obtained in a total running time of 9700 seconds, with the hardest instance (square, n = 6, T = 3, m = 7) requiring about 6266 seconds.

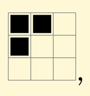
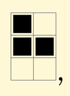
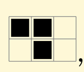
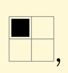
n	T	$\mathcal{G}_T^s(n)$	$\mathcal{G}_T^c(n)$	$\mathcal{G}_T^t(n)$
3	1	0.750	2.000	2.000
	2	0.750	2.000	0.000
	3	0.333	1.000	0.000
4	1	0.800	2.000	2.000
	2	0.750	2.000	1.500
	3	0.800	3.000	1.500
5	1	1.100	2.000	2.000
	2	0.857	2.000	2.000
	3	1.400	4.000	3.000
6	1	1.286	2.000	2.000
	2	1.286	2.000	2.000
	3	1.857	5.000	5.000

Observation 1

We exhibit a family of configurations $X^k \in n(k)^2$, with $n(k) = 3k + 2$, such that

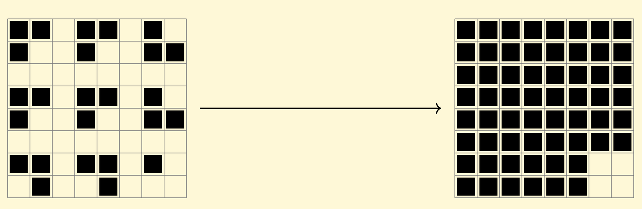
$$g_1^s(X^k) = \frac{6k^2 + 6k - 1}{3k^2 + 6k + 1} \longrightarrow 2 \qquad \text{as } k \to \infty.$$

The construction is based on four small building blocks assembled into a $(3k+2)^2$ grid as follows:

a = , b = , c = , d = ,

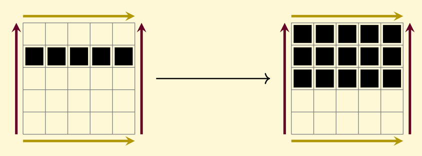
$$X^k = \begin{bmatrix} a & \cdots & a & b \\ \vdots & & \vdots & \vdots \\ a & \cdots & a & b \\ c & \cdots & c & d \end{bmatrix}.$$

A direct count shows that $|X^k| = |a|k^2 + |b|k + |c|k + |d| = 3k^2 + 6k + 1$. After one iteration, every cell is alive except the four cells in the lower-right corner. Hence $|Y^k| = n^2 - 4 = (3k+2)^2 - 4$. An illustration of the case k = 2 is given below.



Observation 2

It holds that $\mathcal{G}_1^c(n), \mathcal{G}_1^t(n) \geq 2$ for all $n \geq 3$. The construction yielding this bound is illustrated below for the toroidal topology with n = 5. The same construction also works for the cylindrical topology.



Lemma

The following inequality holds:

$$\mathcal{G}_1^\star(n) \leq 2 \quad \forall n \geq 3,\ \star \in \{s,c,t\}.$$

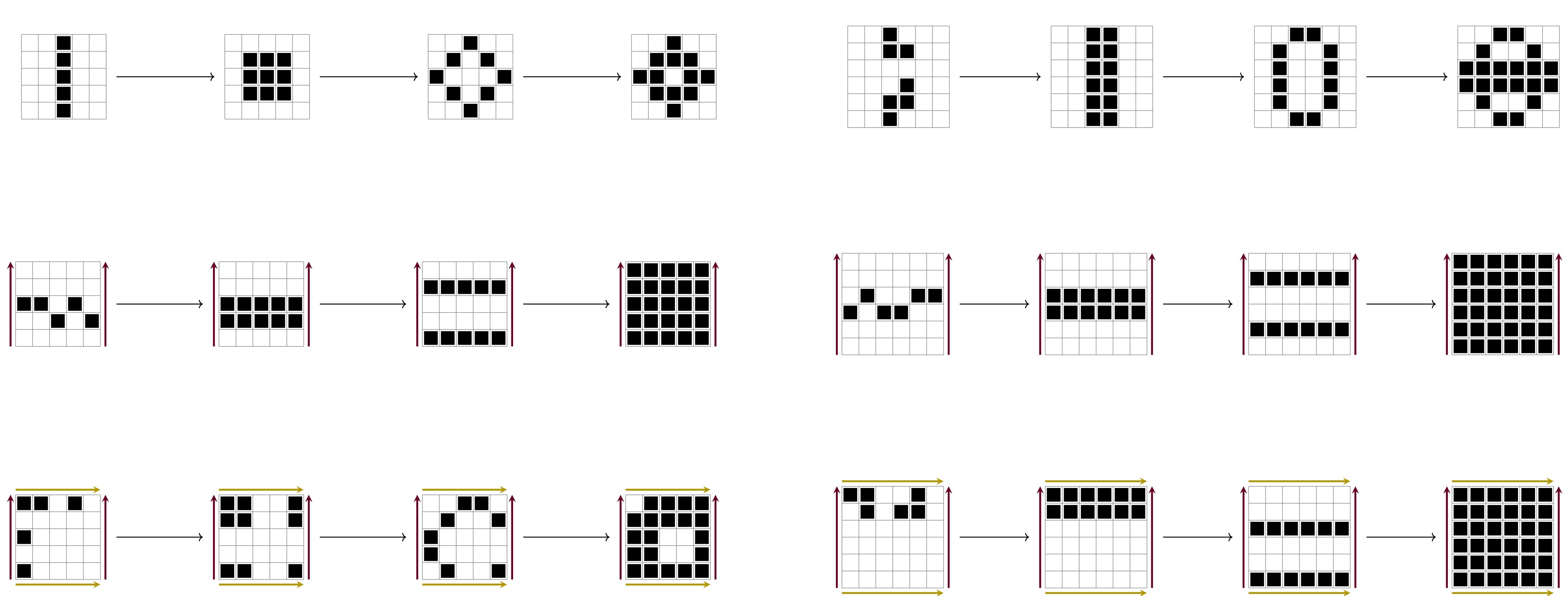
We start with the square topology. Let $X(0) = X$ and $X(1) = Y$. It suffices to prove that $|Y| \leq 3|X|$. Let $\tilde{X}$ be the embedding of X in a grid $\{0, \dots, n+1\}^2$. By a double-counting argument, we have that

$$|X| = \sum_{i,j=1}^{n} X_{ij} = \frac{1}{8} \sum_{i,j=0}^{n+1} \sum_{(k,l)\in a(i,j)} \tilde{X}_{kl} \geq \cdots \geq \frac{3}{8}|Y| - \frac{1}{8}|X| \quad \longrightarrow \quad 8|X| \geq 3|Y| - |X|.$$

For the cylindrical topology, it suffices to embed X in a rectangular grid, with one more row at the top and one more at the bottom, whereas for the toroidal topology, no embedding is needed.

Optimal Solutions for T=3

Shown below are optimal configurations returned by the ILP for n = 5 and n = 6 (square, cylindrical, and toroidal topologies). These examples illustrate spatial structures achieving maximal growth.



Open Questions

By induction, it holds that

$$\mathcal{G}_T^\star(n) \leq (\mathcal{G}_1^\star(n) + 1)^T - 1 = 3^T - 1.$$

Does there exist a function $B(T) < 3^T - 1$ such that $\mathcal{G}_T^\star(n) \leq B(T)$ for all $n \geq 3$ and $\star \in \{s,c,t\}$?

How tight is the LP relaxation, and are there interpretable classes of valid inequalities that significantly improve the formulation?

Takeaway. The maximal one-step growth rate appears to be **bounded by 2** and is achieved under cylindrical and toroidal topologies. Determining the behaviour for larger T and larger grids remains challenging both combinatorially and computationally.

References

Gardner, M., The fantastic combinations of John Conway's new solitaire game 'life'. *Scientific American*, **223**(4), pp. 120–123, 1970.

Bosch, R. A., Integer programming and Conway's game of Life. *SIAM review*, **41**(3), pp. 594–604, 1999.