Generalizing Conway's Game of Life on Graphs

Master's Thesis Proposal, Department of Mathematics

Supervisor: Prof. Stefano Gualandi
Co-supervisor: Ambrogio Maria Bernardelli

1 Introduction

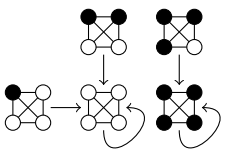
Given an undirected, connected graph $G = (V, E)$ with n nodes and such that every node has degree at least two, we can endow this graph with a Conway's Game of Life (CGoL) initial configuration, that is, an assignment $s : V \to \{0, 1\}^n$. This assignment represents if a node v is "dead" ($s(v) = 0$) or "alive" ($s(v) = 1$). We will denote the graph G endowed with an assignment s with G_s. In particular, G_s is a connected node-colored graph. Given such a graph, we can define the function representing one iteration of CGoL:

$$f(s(v)) = \begin{cases} 1 & \text{if} \quad (s(v) = 0 \ \wedge \ \sum_{u \in N(v)} s(u) = 3) \ \vee \\ & \quad\ (s(v) = 1 \ \wedge \ 2 \leq \sum_{u \in N(v)} s(u) \leq 3), \\ 0 & \text{otherwise,} \end{cases}$$

where $N(v) \coloneqq \{u \in V \mid uv \in E\}$. We will denote by $f(G_s)$ the graph obtained from G_s by the different assignment $f(s)$, that is, $f(G_s)$ represents the configuration that is the CGoL successor of the configuration represented by G_s.

2 Goals of the thesis

Fix a graph $G = (V, E)$, $|V| = n$, let $\hat{\mathcal{G}} \coloneqq \{G_s \mid s \in \{0, 1\}^n\}$, and denote with $\mathcal{G}$ the set $\hat{\mathcal{G}}$ filtered by node-colored graph isomorphism. Define the directed graph $\mathfrak{G} = (\mathcal{G}, A)$ such that $(G_s, G_{s'}) \in A$ iff $G_{s'} = f(G_s)$, that is, connect with an arc two nodes representing two configurations if and only if the endpoint of the arc represents the successor of the startpoint of the arc. For example, every node has only one outgoing arc (counting self-loops), gardens of Eden do not have any ingoing arc, stable configurations have a self-loop. The following is the representation of $\mathfrak{G}$ when $G = K_4$.

$\mathfrak{G} =$  when $G = K_4$.

The goal of this thesis is to study the structure of $\mathfrak{G}$ depending on the initial graph G. Integer programming could be used to answer different questions regarding both the evolution of the configurations and the perturbations on the initial graph.

3 Starting points

Suggested lectures:

- an introduction to graph theory [1];
- an introduction to the game [2].

The first steps could consist in studying the problem

- in the simple case of G being a grid, as in the original CGoL;
- for small values of n;

both from a theoretical and computational point of view.

References

[1] Bollobás, B. (1998). Modern graph theory (Vol. 184). Springer Science & Business Media.

[2] Conway, J. (1970). The game of life. Scientific American, 223(4), 4.