# Coloring of semilattices

Master's Thesis Proposal, Department of Mathematics

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## 1 Introduction

We begin by introducing some basic algebraic structures.

**Definition 1** (Semigroup). *A set $S$ together with a binary operation $\star : S \times S \to S$ is called a* semigroup *if the operation $\star$ is associative, that is, if*

$$(a \star b) \star c = a \star (b \star c) \qquad \forall a, b, c \in S. \tag{1}$$

*We will say that $S$ is* abelian *or* commutative *if the operation $\star$ is symmetric, that is if*

$$a \star b = b \star a \qquad \forall a, b \in S. \tag{2}$$

**Definition 2** (Semilattice). *A commutative semigroup $(S, \star)$ is called a* semilattice *if the binary operation $\star$ is* idempotent*, that is, if*

$$a \star a = a \qquad \forall a \in S. \tag{3}$$

We focus on semilattices endowed with an absorbing element $a_0$, that is an element for which $a_0 \star b = a_0$ for every element $b$ in the semilattice. We will denote this absorbing element simply with 0. From now on, every semilattice we mention will have an absorbing element, if not stated otherwise.

We are interested in studying the following structure.

**Definition 3** (Zero-divisor graph). *Given a semilattice $(S, \star)$, its* zero-divisor graph *$G_S = (S, E_S)$ is a graph with the elements of $S$ as its nodes, with an edge between distinct elements $i, j \in S$ if and only if $i \star j = 0$.*

## 2 Goals of this thesis

This thesis aims to investigate properties of zero-divisor graphs of semilattices, particularly those related to finding a proper coloring through linear programming. Given a zero-divisor graph, we are interested in evaluating the integrality gap of the classical linear formulation of the proper coloring problem, defined as the ratio between the chromatic number $\chi(G)$ and the fractional chromatic number $\chi_f(G)$ of the graph.

Given a graph $G$, the chromatic number $\chi(G)$ is the minimum number of colors needed to assign to the vertices of $G$ so that adjacent vertices receive different colors.

The fractional chromatic number $\chi_f(G)$ is a relaxation of this concept. It allows a "fractional" assignment of colors, typically modeled as a linear program. It can be defined as the solution to the LP relaxation of the integer coloring problem or, equivalently, in terms of covering the graph with independent sets and assigning weights to these sets. It always satisfies $\chi_f(G) \leq \chi(G)$.

The integrality gap $\chi(G)/\chi_f(G)$ measures how far the fractional relaxation is from the integer optimum. See [2, 3] for more details on these concepts.

Studying the coloring properties of such graphs can provide insights into both algebraic structure and combinatorial optimization.

# 3 Starting points

Suggested lectures:

- an introduction to the problem on commutative rings [1];
- an introduction to graph coloring [2];
- an introduction to fractional graph coloring [3].

Initial steps may include studying the following:

- generate all the non-isomorphic semilattices and study their zero-divisor graphs;
- prove theoretical properties of zero-divisor graphs for a particular class of semilattices.

# References

[1] Beck, I. (1988). Coloring of commutative rings. *Journal of algebra*, **116**(1), 208-226.

[2] Jensen, T. R., & Toft, B. (2011). Graph coloring problems. *John Wiley & Sons*.

[3] Scheinerman, E. R., & Ullman, D. H. (2011). Fractional graph theory: a rational approach to the theory of graphs. *Courier Corporation*. (Originally published in 1997)