Cost Structure for Vertices of the Complete Metric formulation of the Steiner Tree Problem

Master's Thesis Proposal, Department of Mathematics

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1 Introduction

Given an undirected, edge-weighted, connected graph $G = (V, E)$ with n nodes and positive costs c_{ij} on each edge $\{i, j\} \in E,\ i, j \in V$, and a subset of nodes $T \subset V$ of cardinality $t \geq 2$, the Steiner Tree Problem (STP) involves finding the minimum-cost tree that spans the set of *terminals* T. The STP is generally modeled and solved via integer linear programming, with many diverse publications released over the years. Among them, the *Bidirected cut formulation* has attracted much attention, thanks to exceptional empirical performances. The core of the formulation consists in replacing each undirected edge $\{i, j\}$ with two arcs (i, j) and (j, i) and introducing a decision variable x_{ij} for each arc. For a given root node $r \in T$, the formulation is presented below:

$$\min_{\mathbf{x} \in \{0,1\}^{2\times|E|}} \quad \sum_{\{i,j\}\in E} c_e(x_{ij} + x_{ji}) \tag{1a}$$

$$\text{s.t.} \quad x_{ij} + x_{ji} \leq 1, \qquad e = \{i, j\} \in E, \tag{1b}$$

$$x\left(\delta^-(W)\right) \geq 1, \qquad W \subset V \setminus \{r\},\ W \cap T \neq \emptyset, \tag{1c}$$

$$x_{ij} \in \{0, 1\}, \tag{1d}$$

where $\delta^-(W) := \{(i, j) \mid i \notin W,\ j \in W\}$.

2 Goals of the thesis

In [1], a novel formulation is proposed, specifically tailored for complete metric graphs. This formulation, called the Complete Metric (CM) formulation, is presented below through its associated polytope

$$P_{CM}(n, t) \coloneqq \{x \in [0, 1]^m \ : \tag{2a}$$

$$x\left(\delta^-(W)\right) \geq 1, \qquad W \subset V \setminus \{r\},\ W \cap T \neq \emptyset, \tag{2b}$$

$$x\left(\delta^-(r)\right) = 0, \tag{2c}$$

$$x\left(\delta^-(v)\right) \leq 1, \qquad v \in V \setminus \{r\}, \tag{2d}$$

$$2x\left(\delta^-(v)\right) \leq x\left(\delta^+(v)\right), \qquad v \in V \setminus T\}. \tag{2e}$$

Additionally, we denote with $S_{CM}(n, t)$ the set of integer points contained in the polytope $P_{CM}(n, t)$.

A nice results regarding these points in presented below.

Theorem 1. *Let* $x \in S_{CM}(n,t)$*. Then,* x *is the unique integer optimal solution for the CM formulation with the metric cost* $c_{ij} = 2 - (x_{ij} + x_{ji}) \in \{1,2\}$*.*

From this theorem, a natural conjecture follows.

Conjecture 1. *Any vertex of* $P_{CM}(n,t)$ *is an optimum for a metric cost* $c_{ij} \in \{1,2\}$*.*

The main goal of the thesis is to prove or disprove this conjecture. Additionally, noteworthy accomplishments include demonstrating it in weaker forms or special cases.

3 Starting points

Suggested lectures:

- an introduction to the problem and its formulations [3, 2];
- the aforementioned theorem and other results on the CM formulation [1].

The first steps could consist in studying the conjecture

- in the simple case of the fractional vertices being pure half-integer, that is, attaining values in the set $\{0, 1/2\}$;
- for the small cases $n = 4, 5$;

both from a theoretical and computational point of view.

References

[1] Bernardelli, A. M., et al., Lower bounds for the integrality gap of the bi-directed cut formulation of the Steiner Tree Problem. *arXiv preprint arXiv:2405.13773*, 2024.

[2] Goemans, M. X., and Myung, Y. S., A catalog of Steiner tree formulations. *Networks*, **23**(1), pp. 19-28, 1993.

[3] Ljubić, I., Solving Steiner trees: Recent advances, challenges, and perspectives, *Networks*, **77**(2), pp. 177-204, 2021.